

MDS Matrices with Lightweight Circuits

Sébastien Duval

Gaëtan Leurent

Sebastien.Duval@inria.fr

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Security of Block Ciphers

Shannon's criteria

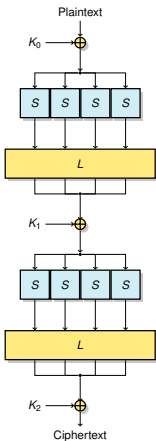
1 Diffusion

- Every bit of plaintext and key must affect every bit of the output
- We usually use **linear** functions

2 Confusion

- Relation between plaintext and ciphertext must be intractable
- Requires **non-linear** operations
- Often implemented with tables: **S-Boxes**

SPN Ciphers



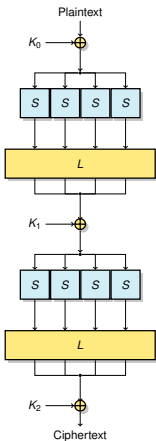
Differential Branch Number

$$B_d(L) = \min_{x \neq 0} \{w(x) + w(L(x))\}$$

Linear Branch Number

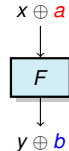
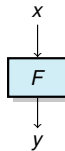
$$B_l(L) = \min_{x \neq 0} \{w(x) + w(L^\top(x))\}$$

SPN Ciphers

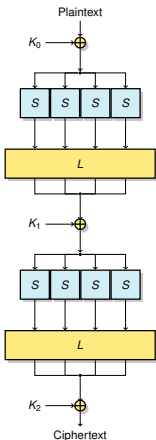


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SPN Ciphers



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$$B_d(L) = \min_{x \neq 0} \{w(x) + w(L(x))\}$$

Linear Branch Number

$$B_l(L) = \min_{x \neq 0} \{w(x) + w(L^T(x))\}$$

Maximum branch number : $k + 1$
 Can be obtained from MDS codes

Diffusion Matrices

$$\begin{bmatrix} 2 & 3 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 3 \\ 3 & 1 & 1 & 2 \end{bmatrix}$$

Usually on finite fields:

x a primitive element of \mathbb{F}_{2^n}

$2 \leftrightarrow x$

$3 \leftrightarrow x + 1$

Coeffs. = polynomials in x with binary coefficients

i.e. coeffs. $\in \mathbb{F}_2[x]/P$, with P a primitive polynomial

Characterization

L is MDS iff its minors are non-zero

Going Lightweight

lightweight cipher = lightweight S-Boxes + lightweight diffusion matrix

Focus on the diffusion function

Goal: Find lightweight MDS matrix

Main approaches:

- ▶ Optimize existing ciphers: MDS matrix \rightarrow reduce cost (AES MixColumns)
- ▶ New ciphers: lightweight by design

Previous Works

Recursive Matrices

Guo, Peyrin and Poschmann in PHOTON (used in LED)

A lightweight matrix

A^i MDS

Implement A , then iterate A i times.

Optimizing Coefficients

- ▶ Structured matrices: restrict to a small subspace with many MDS matrices
- ▶ More general than finite fields: less costly operations than multiplication in a finite field

Cost Evaluation

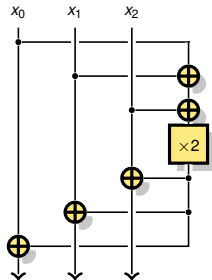
Previous work: Number of XORS + sum of cost of each coefficient

Drawback: Cannot reuse intermediate values

Our approach: Global optimization as a circuit

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}$$

Previous: $\begin{cases} 6 \text{ mult. by } 2 \\ 3 \text{ mult. by } 3 \\ 6 \text{ XORS} \end{cases}$

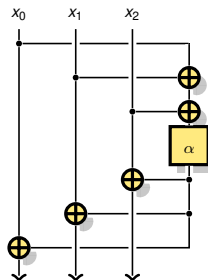


New: $\begin{cases} 1 \text{ mult. by } 2 \\ 5 \text{ XORS} \end{cases}$

Formal Matrices

Finite fields \rightarrow polynomial ring

- ▶ α linear mapping on \mathbb{F}_{2^n}
- ▶ Coefficients $\in \mathbb{F}_2[\alpha]$
i.e. polynomials in α with
coeffs. in \mathbb{F}_2



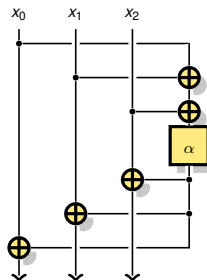
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Formal matrices

- ▶ α undefined
- \Rightarrow formal coefficients/matrix
- ▶ Objective: find $M(\alpha)$ s.t.
 $\exists A, M(A)$ MDS



MDS Characterization of Formal Matrices

MDS Characterization

Maximal branch number iff the minors are non-zero (call it *formal MDS*)

Caution: minors are polynomials in α

$M(\alpha)$ formal MDS $\Leftrightarrow \exists A, M(A)$ MDS

Objective

- ▶ Find $M(\alpha)$ formal MDS and lightweight
- ▶ Fix n
- ▶ Find A linear mapping over \mathbb{F}_{2^n} lightweight s.t. $M(A)$ MDS

Algorithm

Exhaustive search over circuits

Search Space

MDS matrices of sizes 3×3 and 4×4

For **any** word size n

Operations:

- ▶ word-wise XOR
- ▶ α (generalization of a multiplication)
- ▶ Copy

r registers: one register per word (3 for 3×3)

+ (at least) one more register \rightarrow more complex operations

Very costly

Implementation: Main Idea

Graph-based search

- ▶ Node = matrix = sequence of operations
- ▶ Lightest implementation = shortest path to MDS matrix
- ▶ When we spawn a node, we test if it is MDS

Representation

$k \times r$ matrix, coefficients are polynomials in $\mathbb{F}_2[\alpha]$

Optimizations: Cut Useless Branches

Limit use of Copy

After copy, force use of the copied value

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Set up Boundaries

Choose maximum cost and maximum depth for circuits

+ many more optimizations to save memory (at the cost of computation time)

Optimizations: A^*

A^*

Idea of A^*

- ▶ Guided Dijkstra
- ▶ weight = weight from origin + estimated weight to objective

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- ▶ Heuristic
- ▶ How far from MDS ?

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- ▶ Heuristic
- ▶ How far from MDS ?
- ▶ Column with a 0: cannot be part of MDS matrix
- ▶ Linearly dependent columns: not part of MDS matrix
- ▶ Estimate: $m = \text{rank of the matrix (without columns containing 0)}$
- ▶ Need at least $k - m$ word-wise XORs to MDS

Result: much faster

Optimizations: Use Equivalence

- ▶ `TestedNodes`: list of all nodes that have been tested for MDS
- ▶ `UntestedNodes`: list of all untested nodes

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- ▶ `MDS? false` \rightarrow spawn all children nodes in `UntestedNodes`

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- ▶ Add M to `TestedNodes`

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Use Equivalence

Matrices are equivalent up to reordering of input/output words

Use unique ID for equivalent nodes

Store `TestedIDs` rather than `TestedNodes`

Extensions

Additional Read-only Registers

Allow for use of the input values of the function at any time

Inverse

Allow use of α^{-1}

Powers

Allow use of α^2

Independent Operations

Allow use of 3 independent linear operations α, β, γ

3 × 3 MDS Search

Depth	Cost	Extensions	Memory
4	5 XOR, 1 LIN		14
3	5 XOR, 2 LIN		5
2	6 XOR, 3 LIN	RO_IN	4

Table: Optimal 3×3 MDS matrices (all results are obtained in less than 1 second, memory is given in MB).

3 × 3 MDS Matrices

Depth	Cost	M	Fig.
4	5 XOR, 1 LIN	$M_{3,4}^{5,1} = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}$	
		$M_{3,4}^{5,1'} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix}$	

3 × 3 MDS Matrices

Depth	Cost	M	Fig.
3	5 XOR, 2 LIN	$M_{3,3}^{5,2} = \begin{bmatrix} 3 & 1 & 3 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$	
2	6 XOR, 3 LIN	$M_{3,2}^{6,3} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$	

4 × 4 MDS Matrices

Depth	Cost	Extensions	Memory (GB)	Time (h)
6	8 XOR, 3 LIN		30.9	19.5
5	8 XOR, 3 LIN	INDEP	24.3	2.3
5	9 XOR, 3 LIN		154.5	25.6
4	8 XOR, 4 LIN	MAX_POW = 2	274	30.2
4	9 XOR, 3 LIN	INDEP	46	4.5
4	9 XOR, 4 LIN		77.7	12.8
3	9 XOR, 5 LIN	INV	279.1	38.5

Table: Optimal 4 × 4 MDS matrices.

4 × 4 MDS Matrices

Depth	Cost	M	Fig.
6	8 XOR, 3 LIN	$M_{4,6}^{8,3} = \begin{bmatrix} 3 & 1 & 4 & 4 \\ 1 & 3 & 6 & 4 \\ 2 & 2 & 3 & 1 \\ 3 & 2 & 1 & 3 \end{bmatrix}$	

4 × 4 MDS Matrices

Depth	Cost	M	Fig.
5	8 XOR, 3 LIN	$M_{4,5}^{8,3} = \begin{bmatrix} \alpha + \gamma & \alpha & \gamma & \gamma \\ \alpha + \gamma + 1 & \alpha + 1 & \gamma + 1 & \gamma \\ 1 & 1 & \beta + 1 & \beta \\ \gamma + 1 & 1 & \beta + \gamma + 1 & \beta + \gamma \end{bmatrix}$	
5	9 XOR, 3 LIN	$M_{4,5}^{9,3} = \begin{bmatrix} 2 & 2 & 3 & 1 \\ 1 & 3 & 6 & 4 \\ 3 & 1 & 4 & 4 \\ 3 & 2 & 1 & 3 \end{bmatrix}$	

4 × 4 MDS Matrices

Depth	Cost	M	Fig.
4	8 XOR, 4 LIN	$M_{4,4}^{8,4} = \begin{bmatrix} 5 & 7 & 1 & 3 \\ 4 & 6 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 1 & 1 & 4 & 6 \end{bmatrix}$	
		$M_{4,4}^{8,4'} = \begin{bmatrix} 6 & 7 & 1 & 5 \\ 2 & 3 & 1 & 1 \\ 1 & 5 & 6 & 7 \\ 1 & 1 & 2 & 3 \end{bmatrix}$	

4 × 4 MDS Matrices

Depth	Cost	M	Fig.
4	9 XOR, 3 LIN	$M_{4,4}^{9,3} = \begin{bmatrix} \alpha + 1 & \alpha & \gamma + 1 & \gamma + 1 \\ \beta & \beta + 1 & 1 & \beta \\ 1 & 1 & \gamma & \gamma + 1 \\ \alpha & \alpha + 1 & \gamma + 1 & \gamma \end{bmatrix}$	
4	9 XOR, 4 LIN	$M_{4,4}^{9,4} = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 3 & 2 & 3 \\ 3 & 3 & 5 & 1 \\ 3 & 1 & 1 & 3 \end{bmatrix}$	

4 × 4 MDS Matrices

Depth	Cost	M	Fig.
3	9 XOR, 5 LIN	$M_{4,3}^{9,5} = \begin{bmatrix} \alpha + \alpha^{-1} & \alpha & 1 & 1 \\ 1 & \alpha + 1 & \alpha & \alpha^{-1} \\ 1 + \alpha^{-1} & 1 & 1 & 1 + \alpha^{-1} \\ \alpha^{-1} & \alpha^{-1} & 1 + \alpha^{-1} & 1 \end{bmatrix}$	

From Formal Matrices to Instances

The Idea

- 1 Input: Formal matrix $M(\alpha)$ MDS
- 2 Output: $M(A)$ MDS, with A a linear mapping (the lightest we can find)

Characterization of MDS Instantiations

MDS Test

▶ Intuitive approach:

- 1 Choose A a linear mapping
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► We can start by computing the minors:

- 1 Let I, J subsets of the lines and columns
- 2 Define $m_{I,J} = \det_{\mathbb{F}_2[\alpha]}(M_{I,J})$
- 3 $M(A)$ is MDS iff all $m_{I,J}(A)$ are non-zero

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 - 3 $M(A)$ is MDS iff all $m_{I,J}(A)$ are non-zero
- ▶ With the minimal polynomial
 - 1 Let μ_A the minimal polynomial of A
 - 2 $M(A)$ is MDS iff $\forall(I, J), \gcd(\mu_A, m_{I,J}) = 1$

General Idea of Instantiation

We want A s.t. $\forall(I, J), \gcd(\mu_A, m_{I,J}) = 1$

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Low Cost Instantiation

- ▶ Pick π with few coefficients: a trinomial requires 1 rotation + 1 binary xor
- ▶ If using A^{-1} or A^2 , make sure they are lightweight too

Concrete Choices of A

We need to fix the size

Branches of size 4 bits (\mathbb{F}_{2^4})

$$A_4 = \begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 1 & 1 & \cdot & \cdot \end{bmatrix}$$

(companion matrix of $X^4 + X + 1$ (irreducible))

$$A_4^{-1} = \begin{bmatrix} 1 & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{bmatrix}$$

(minimal polynomial is $X^4 + X^3 + 1$)

Branches of size 8 bits (\mathbb{F}_{2^8})

$$A_8 = \begin{bmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

(companion matrix of $X^8 + X^2 + 1 = (X^4 + X + 1)^2$)

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(minimal polynomial is $X^8 + X^6 + 1$)

Example of Instantiation: \mathbb{F}_{2^8}

In \mathbb{F}_2^8 , the trinomials and their factorization are

$$X^8 + X + 1 = (X^2 + X + 1)(X^6 + X^5 + X^3 + X^2 + 1),$$

$$X^8 + X^2 + 1 = (X^4 + X + 1)^2,$$

$$X^8 + X^3 + 1 = (X^3 + X + 1)(X^5 + X^3 + X^2 + X + 1),$$

$$X^8 + X^4 + 1 = (X^2 + X + 1)^4,$$

$$X^8 + X^5 + 1 = (X^3 + X^2 + 1)(X^5 + X^4 + X^3 + X^2 + 1),$$

$$X^8 + X^6 + 1 = (X^4 + X^3 + 1)^2,$$

$$X^8 + X^7 + 1 = (X^2 + X + 1)(X^6 + X^4 + X^3 + X + 1).$$

In particular, there are only 2 trinomials which factorize to degree 4 polynomials: $X^8 + X^2 + 1 = (X^4 + X + 1)^2$ and $X^8 + X^6 + 1 = (X^4 + X^3 + 1)^2$.

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$$X^8 + X^4 + 1 = (X^2 + X + 1)^4,$$

$$X^8 + X^5 + 1 = (X^3 + X^2 + 1)(X^5 + X^4 + X^3 + X^2 + 1),$$

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Example of Instantiation: $M_{4,6}^{8,3}$

The minors of $M_{4,6}^{8,3} = \begin{bmatrix} 2 & 2 & 3 & 1 \\ 1 & 3 & 6 & 4 \\ 3 & 1 & 4 & 4 \\ 3 & 2 & 1 & 3 \end{bmatrix}$ are

$\{1, X, X + 1, X^2, X^2 + 1, X^2 + X, X^2 + X + 1, X^3, X^3 + 1, X^3 + X, X^3 + X + 1, X^3 + X^2 + 1, X^3 + X^2 + X, X^3 + X^2 + X + 1\}$

whose factors are

$$\{X, X + 1, X^3 + X + 1, X^2 + X + 1, X^3 + X^2 + 1\}$$

On 4 bits: Degrees $\leq 3 \Rightarrow$ relatively prime with $X^4 + X + 1$ and $X^4 + X^3 + 1$ because irreducible

$$\alpha = A_4 \text{ or } \alpha = A_4^{-1} \Rightarrow \text{MDS matrix over } \mathbb{F}_{2^4}.$$

On 8 bits: All relatively prime with $X^8 + X^2 + 1$ and $X^8 + X^6 + 1$ ($(X^4 + X + 1)^2$ and $(X^4 + X^3 + 1)^2$)

$$\alpha = A_8 \text{ or } \alpha = A_8^{-1} \Rightarrow \text{MDS matrix over } \mathbb{F}_{2^8}.$$

Example of Instantiation: $M_{4,4}^{8,4}$

The factors of the minors of $M_{4,4}^{8,4} = \begin{bmatrix} 5 & 7 & 1 & 3 \\ 4 & 6 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 1 & 1 & 4 & 6 \end{bmatrix}$ are

$$\{X, X + 1, X^3 + X + 1, X^2 + X + 1, X^3 + X^2 + 1, X^4 + X^3 + 1\}$$

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Factors of degree ≤ 3 relatively prime with $X^8 + X^2 + 1$ and $X^8 + X^6 + 1$.

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Factors of degree ≤ 3 relatively prime with $X^8 + X^2 + 1$ and $X^8 + X^6 + 1$.

On 4 bits: Not relatively prime with $X^4 + X^3 + 1$ but all relatively prime with $X^4 + X + 1$.

$\alpha = A_4 \Rightarrow$ MDS matrix over \mathbb{F}_{2^4} .

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$$\{X, X + 1, X^3 + X + 1, X^2 + X + 1, X^3 + X^2 + 1, X^4 + X^3 + 1\}$$

Factors of degree ≤ 3 relatively prime with $X^8 + X^2 + 1$ and $X^8 + X^6 + 1$.

On 4 bits: Not relatively prime with $X^4 + X^3 + 1$ but all relatively prime with $X^4 + X + 1$.

$\alpha = A_4 \Rightarrow$ MDS matrix over \mathbb{F}_{2^4} .

On 8 bits: Not relatively prime with $X^8 + X^6 + 1$ but all relatively prime with $X^8 + X^2 + 1$.

$\alpha = A_8 \Rightarrow$ MDS matrix over \mathbb{F}_{2^8} .

Comparison With Existing MDS Matrices

Size	Ring	Matrix	Cost			Ref
			Naive	Best	Depth	
$M_4(M_8(\mathbb{F}_2))$	$GL(8, \mathbb{F}_2)$	Circulant	106			(Li Wang 2016)
	$GL(8, \mathbb{F}_2)$	Hadamard		72	6	(Kranz <i>et al.</i> 2018)
	$\mathbb{F}_2[\alpha]$	$M_{4,6}^{8,3}$		67	6	$\alpha = A_8$ or A_8^{-1}
	$\mathbb{F}_2[\alpha]$	$M_{4,5}^{8,3}$		68	5	$\alpha = A_8, \beta = A_8^{-1}, \gamma = A_8^{-2}$
	$\mathbb{F}_2[\alpha]$	$M_{4,4}^{8,4}$		70	4	$\alpha = A_8$
	$\mathbb{F}_2[\alpha]$	$M_{4,3}^{9,5}$		77	3	$\alpha = A_8$ or A_8^{-1}
$M_4(M_4(\mathbb{F}_2))$	$GF(2^4)$	$M_{4,n,4}$	58	58	3	(Jean Peyrin Sim 2017)
	$GF(2^4)$	Toeplitz	58	58	3	(Sarkar Syed 2016)
	$GL(4, \mathbb{F}_2)$	Subfield		36	6	(Kranz <i>et al.</i> 2018)
	$\mathbb{F}_2[\alpha]$	$M_{4,6}^{8,3}$		35	6	$\alpha = A_4$ or A_4^{-1}
	$\mathbb{F}_2[\alpha]$	$M_{4,5}^{8,3^{-1}}$		36	5	$\alpha = A_4, \beta = A_4^{-1}, \gamma = A_4^{-2}$
	$\mathbb{F}_2[\alpha]$	$M_{4,4}^{8,4}$		38	4	$\alpha = A_4$
	$\mathbb{F}_2[\alpha]$	$M_{4,3}^{9,5}$		41	3	$\alpha = A_4$ or A_4^{-1}